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LETTER TO THE EDITOR

Laplace transforms of Airy functions

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Abstract. Analytic expressions, involving gamma, incomplete gamma and degenerate hypergeometric functions, are obtained for the Laplace transforms of Airy functions, which arise in the theory of electron tunnelling from solids.

When an electron is subjected to an electric field, it experiences a linear potential. For such a potential, the Schrödinger equation becomes the Airy equation, whose solutions are well known (Lebedev 1965). In field-emission experiments (Gadzuk and Plummer 1973), an electric field is applied to a crystal surface, and the linear potential barrier so formed allows electrons to tunnel out from the bulk and surface states of the crystal. This process may be analysed by means of the sudden approximation of time-dependent perturbation theory (Schiff 1968), and the tunnelling current obtained.

In performing these calculations, the need arises to evaluate integrals of the form

$$I_{\mathrm{F}}(p)=\int_0^\infty \mathrm{e}^{-pz}\,\mathrm{Fi}(-z)\,\mathrm{d}z \tag{1}$$

where

$$\mathrm{Fi}(-z)=\mathrm{Ai}(-z),\mathrm{Bi}(-z) \tag{2}$$

satisfies the Airy equation

$$\mathrm{Fi}''(-z)+z\,\mathrm{Fi}(-z)=0. \tag{3}$$

The purpose of this letter is to solve the integrals in (1), which are the Laplace transforms of the respective Airy functions, by using (3).

Integrating (1) successively by parts leads to

$$I_{\mathrm{F}}(p)=p^{-1}\,\mathrm{Fi}(0)+p^{-2}\,\mathrm{Fi}'(0)+p^{-2}\int_0^\infty \mathrm{e}^{-pz}\,\mathrm{Fi}''(-z)\,\mathrm{d}z$$

which by (3) becomes

$$I_{\mathrm{F}}(p)=p^{-1}\,\mathrm{Fi}(0)+p^{-2}\,\mathrm{Fi}'(0)-p^{-2}\int_0^\infty z\,\mathrm{e}^{-pz}\,\mathrm{Fi}(-z)\,\mathrm{d}z$$

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i.e.

$$-I'_{\mathrm{F}}(p)+p^2I_{\mathrm{F}}(p)=p\ \mathrm{Fi}(0)+\mathrm{Fi}'(0)$$

or

$$\frac{\mathrm{d}}{\mathrm{d}p}[-I_{\mathrm{F}}(p)\exp(-p^3/3)]=\exp(-p^3/3)[p\ \mathrm{Fi}(0)+\mathrm{Fi}'(0)].$$

Integrating between 0 and p gives

$$I_{\mathrm{F}}(p)=\exp(p^3/3)\left(I_{\mathrm{F}}(0)-\int_0^p \exp(-p^3/3)[p\ \mathrm{Fi}(0)+\mathrm{Fi}'(0)]\,\mathrm{d}p\right)$$

which, on setting $t=p^3/3$, yields

$$I_{\mathrm{F}}(p)=\exp(p^3/3)\left(I_{\mathrm{F}}(0)-3^{-1/3}\,\mathrm{Fi}(0)\int_0^{p^3/3} \mathrm{e}^{-t}t^{-1/3}\,\mathrm{d}t-3^{-2/3}\,\mathrm{Fi}'(0)\int_0^{p^3/3}\mathrm{e}^{-t}t^{-2/3}\,\mathrm{d}t\right). \tag{4}$$

However,

$$\int_0^b \mathrm{e}^{-t}t^{a-1}\,\mathrm{d}t=\gamma(a,b) \tag{5}$$

γ being the incomplete gamma function (Abramowitz and Stegun 1964). Thus, (5) in (4) gives

$$I_{\mathrm{F}}(p)=\exp(p^3/3)[I_{\mathrm{F}}(0)-3^{-1/3}\,\mathrm{Fi}(0)\gamma(\tfrac{2}{3},p^3/3)-3^{-2/3}\,\mathrm{Fi}'(0)\gamma(\tfrac{1}{3},p^3/3)]. \tag{6}$$

In view of (2), equation (6) leads to (Abramowitz and Stegun 1964)

$$I_{\mathrm{A}}(p)=3^{-1}\exp(p^3/3)[2-\gamma(\tfrac{2}{3},p^3/3)/\Gamma(\tfrac{2}{3})+\gamma(\tfrac{1}{3},p^3/3)/\Gamma(\tfrac{1}{3})] \tag{7}$$

and

$$I_{\mathrm{B}}(p)=3^{-1/2}\exp(p^3/3)[\gamma(\tfrac{2}{3},p^3/3)/\Gamma(\tfrac{2}{3})+\gamma(\tfrac{1}{3},p^3/3)/\Gamma(\tfrac{1}{3})] \tag{8}$$

via

$$I_{\mathrm{A}}(0)=\int_0^\infty \mathrm{Ai}(\pm z)\,\mathrm{d}z=\tfrac{1}{3},\tfrac{2}{3},\qquad I_{\mathrm{B}}(0)=\int_0^\infty \mathrm{Bi}(-z)\,\mathrm{d}z=0, \tag{9}$$

and

$$\mathrm{Ai}(0)=3^{-1/2}\,\mathrm{Bi}(0)=[3^{2/3}\Gamma(\tfrac{2}{3})]^{-1},\qquad -\mathrm{Ai}'(0)=3^{-1/2}\,\mathrm{Bi}'(0)=[3^{1/3}\Gamma(\tfrac{1}{3})]^{-1}. \tag{10}$$

Laplace transforms of the type

$$\Phi_{\mathrm{A}}(p)=\int_0^\infty \mathrm{e}^{-pz}\,\mathrm{Ai}(z)\,\mathrm{d}z \tag{11}$$

where $\mathrm{Ai}(z)$ now satisfies

$$\mathrm{Ai}''(z)-z\ \mathrm{Ai}(z)=0 \tag{12}$$

may also be treated by the above procedure. In this case, (4) is replaced by (Gradshteyn and Ryzhik 1965)

$$\Phi_{\mathrm{A}}(b)=\mathrm{e}^{-b}[\Phi_{\mathrm{A}}(0)+3^{-1/3}\,\mathrm{Ai}(0)\phi(\tfrac{1}{3},b)+3^{-2/3}\,\mathrm{Ai}'(0)\phi(\tfrac{2}{3},b)] \tag{13}$$

where

$$b = p^3/3 \tag{14}$$

and

$$\phi(a, b) = \int_0^b e^t t^{-a}\, dt = b^{1-a}(1-a)^{-1}{}_1F_1(1-a, 2-a; b) \qquad (15), (16)$$

${}_1F_1$ being the degenerate hypergeometric function. Hence, with the aid of (9) and (10), substituting (16) in (13) yields

$$\Phi_A(b) = e^{-b}[\tfrac{1}{3} + b^{2/3}{}_1F_1(\tfrac{2}{3}, \tfrac{5}{3}; b)/2\Gamma(\tfrac{2}{3}) - b^{1/3}{}_1F_1(\tfrac{1}{3}, \tfrac{4}{3}; b)/\Gamma(\tfrac{1}{3})]. \tag{17}$$

Finally, it is interesting to note that, in the integrand of (15),

$$t^{-a} = [\Gamma(a)]^{-1} \int_0^\infty u^{a-1}\, e^{-tu}\, du \tag{18}$$

from the integral definition of the gamma function. Inserting (18) in (15), and reversing the order of integration, gives (Gradshteyn and Ryzhik 1965)

$$\phi(a, b) = [\Gamma(a)]^{-1} \int_0^\infty \int_0^b u^{a-1}\, e^{t(1-u)}\, dt\, du = [\Gamma(a)]^{-1}[M(a) - e^b L(a, b)] \tag{19}$$

where

$$M(a) = \int_0^\infty u^{a-1}(u-1)^{-1}\, du = -\pi \cot(\pi a) \tag{20}$$

and

$$L(a, b) = \int_0^\infty e^{-bu} u^{a-1}(u-1)^{-1}\, du. \tag{21}$$

Thus, equating (16) and (19) enables the evaluation

$$L(a, b) = e^{-b}[M(a) - b^{1-a}(1-a)^{-1}\Gamma(a){}_1F_1(1-a, 2-a; b)] \tag{22}$$

to be achieved, which augments the solution (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty e^{-bu} u^{a-1}(u+\alpha)^{-1}\, du = \alpha^{a-1}\, e^{\alpha b}\Gamma(a)\Gamma(1-a, \alpha b) \tag{23}$$

that requires $|\arg \alpha| < \pi$, Re $b > 0$ and Re $a > 0$.

Note added in proof. After completing this article, the authors became aware of other work on this problem (Smith 1973).

References

Abramowitz M and Stegun I A 1964 (revised 1972) *Handbook of Mathematical Functions* (US Dept of Commerce, National Bureau of Standards, Washington DC *Appl. Math. Ser.* 55)

Gadzuk J W and Plummer E W 1973 *Rev. Mod. Phys.* **45** 487

Gradshteyn I S and Ryzhik I M 1965 *Tables of Integrals, Series and Products* (New York: Academic)

Lebedev N N 1965 *Special Functions and Their Applications* (Englewood Cliffs: Prentice-Hall)

Schiff L I 1968 *Quantum Mechanics* (New York: McGraw-Hill)

Smith P 1973 *SIAM Rev.* **15** 796